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Gianni Ricci (Ed.)

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INTRODUCTION

This book contains a selection of the papers presented at the symposium on "Decision processes in Economics" which was held in Modena (Italy) on 9-10 October 1989.

It coincided with the annual meeting of the italian group on Game Theory; the group is formed by economists, mathematicians, engineers and social scientists.

One of the targets of the Meeting, and therefore of the book, is to create an opportunity for having together papers by scientists with an "optimal control" education and papers by theorists on refinement of equilibrium, on repeted games and other topics. These two modes of working on Games are quite different but we think that a unitary approch to Games can be given and this book is an attempt in this direction. Another important and updated issue which is emphisized in the book is the discussion of computation and efficiency of numerical methods in Games.

Stochastic differential games are treated in the papers by Basar, Haurie -and Deissemberg.

Basar considers a stochastic model of a conflict situation between the monetary policy maker (government) and the responding agent (private sector). Because of asymmetry in the (stochastic) information available the Nash and the Stackelberg games become non standard stochastic differential games. After the discussion of the conditions leading to a solution he provides a numerical example for the proposed game.

Haurie considers a game where the observed state changes according to a stochastic jump process. Between two succesive random jump times the process can be either fully deterministic, as in the example of duopoly and innovation or represented as a diffusion process as in the example of fisheres exploitation

The third paper on Stochastic Games is by Deissemberg; he starts observing that each model is an approximation of reality and that the actual policies can increase the model uncertainty. One has to think that the true game lies within a bounded class of games. In discrete control setting Leitmann and Wan proposed a Lyapunov minimax approch that in Deissemberg is extended to continuous-time, non cooperative games.

The papers by Bassetti and Ricci deal with applications of bargaining games.

The first tries to give a new microfoundation to the portfolio selection problem imbedding the decision process into a cooperative game where the two players are identified with the two personalities in which the investor split himself.

The second considers the well known 1967 Goodwin model and determines the Nash bargaining solution of a Game in which capitalists and workers cooperate in eliminating fluctuations in workers' employment rate and in capitalists' profits.

The computation of a solution through a modified decomposition scheme of Dantzig and Wolfe is presented in the paper by Vial.

Another paper which can be usefull in practice is the one by Cavazzuti. Given a game in normal form with a unique Nash point he describes how to define a sequence of approximations converging to the solution of the game.

The papers by Pederzoli and Gambarelli deal with formation and performance of coalitions in n-person games. The first investigates the effect of communication while the second author is more interested in power indices of coalition and on the weights of the corresponding members. They both give indications of possible applications in Social Science, Finance and Politics.
Vannucci is interested in social choices and in his paper presents an analysis of capacities of simple games w.r.t. various alternative noncooperative solution concepts.
Gruber, in the framework of infinitely repeated games, shows how diffusion of quality innovation is an equilibrium outcome of firms competing a' la Cournot and what are the conditions for having always the same firm to innovate first, i.e. what generates persistence of leadership.
The two papers by Battigalli and Gilli deal with extensive games.
The first presents an iterative procedure for obtaining an algorithmic solution under rationality assumption for the players.
The second analyse the concept of correlated rationalizability from Bayesian viewpoint. In particular he studies the relationships among correlated rationalizability, its refinements, undominated stategies and uniqueness.
Finally the paper by Mori shows that information asymmetry have a precise role in the emergence of trade unions. He adopt Harsanyi and Selten's generalization of the Nash solution to bargaining games with incomplete information to prove that it is always advantageous for the workers to delegate a union to bargain with the firm on their behalf.

GIANNI RICCI
Modena, 1990

A CONTINUOUS-TIME MODEL OF MONETARY POLICY AND INFLATION:
A STOCHASTIC DIFFERENTIAL GAME

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1 Introduction

Cuckierman and Meltzer (1986) and Başar and Salmon (1988) have recently studied, in the discrete time, a stochastic model of monetary policy and inflation, that incorporates asymmetric information between the private sector and the monetary authority. In this model, the private sector is a passive player who simply forms conditional (rational) expectations of the current inflation rate, which constitutes the surprise component of the policy maker's (the monetary authority) objective function. The policy maker strives to maximize the objective function by choosing a control policy which also affects the information carried to the passive player whose rational expectations in turn influence the performance of that policy. Under an asymmetric mode of decision making, the model leads to a stochastic control problem of the nonneutral type, whose solution involves active learning. The complete solution to this nonstandard dynamic optimization problem has been obtained by Başar (1988), by developing an indirect approach that necessitates the introduction of a sequence of zero-sum games. This solution corresponds to a Stackelberg solution for the original game, where the monetary authority is the leader. If instead, the roles of the decision makers are taken to be symmetric, while still retaining the asymmetry in the information structure, we obtain the Nash equilibrium solution, which is the one that was discussed by Cuckierman and Meltzer (1986); see also Başar and Salmon (1989) for a discussion of that point.

In this paper we introduce a continuous-time counterpart of the above model, and obtain the best monetary policy under the Stackelberg and Nash equilibrium concepts. We also present some numerical results to complement the theoretical study.

2 Problem Formulation

Here we formulate a continuous-time version of discrete-time model considered earlier by Cuckierman and Meltzer (1986), and Başar and Salmon (1988). It is a two-player differential game model of a "conflict" situation between the monetary policy maker (government) and the responding agent (private sector). The scalar state equation describes the evolution of the *preference parameter* of the monetary policy maker (P):

$$dx_t = -(1-\rho)x_t dt + A(1-\rho)dt + dv_t \tag{1}$$

where $\rho \epsilon (0,1)$, $A > 0$, $\{v_t\}_{t\geq 0}$ is a Wiener process with $\text{var}(dv_t) = \sigma_v^2 dt$, and $x_0 \sim N(\bar{x}_0, \sigma_0^2)$ which is independent of $\{v_t\}$. The control variable for P is the planned rate of monetary growth, m_t^P, leading to the actual cumulative monetary growth, m_t, through the equation

$$\begin{aligned} & dm_t = m_t^P dt + dw_t, \qquad m_0 = 0 \\ \Leftrightarrow & \\ & m_t = \textstyle\int_0^t m_s^P ds + w_t \end{aligned} \tag{2}$$

where $\{w_t\}_{t\geq 0}$ is a Wiener process with variance $\text{var}(dw_t) = \sigma_w^2 dt$, and independent of x_0 and $\{w_t\}$. Furthermore

$$m_t^P = \gamma_t(x_s, s \leq t; m_\tau, \tau < t), \quad t \geq 0 \tag{3}$$

which says that the control variable m_t^P is adapted to the information fields generated by $\{x_s\}_{0\leq s\leq t}$ and $\{m_s\}_{0\leq s<t}$, with $\{\gamma_t\}_{t\geq 0} \in \Gamma$, where Γ is an appropriate strategy space for P.

The responding agent (A) attempts to predict the planned rate of monetary growth, m_t^P, by observing the past history of cumulative monetary growth, $\{m_s\}_{0\leq s<t}$. Denoting the policy variable of A by δ, and the actual predicted value (control variable) at time t by m_t^A, we have

$$m_t^A = \delta_t(m_s, s < t); \qquad \{\delta_t\}_{t\geq 0} \in \Delta \tag{4}$$

where Δ is an appropriate space of policies. A's objective function (to be minimized) is naturally

$$J^A(\gamma,\delta) = E\left\{\int_0^{t_f} (m_t^P - m_t^A)^2 e^{-\tilde{\beta}t} dt\right\} \tag{5}$$

where $[0, t_f]$ is the decision interval, and $\tilde{\beta} \geq 0$ is an appropriate discount factor. Note that the dependence on γ in J^A is through $\{m_t\}_{t\geq 0}$, both directly and also through δ.

The objective function of P, on the other hand, is

$$J^P(\gamma,\delta) = E\left\{\int_0^{t_f} \left[x_t(m_t^P - m_t^A) - \frac{1}{2}(m_t^P)^2\right] e^{-\beta t} dt\right\} \tag{6}$$

which is to be maximized. This reflects a certain tradeoff between maximization of the first term (monetary surprise) and minimization of $(m_t^P)^2$ (low level of inflation), with the preference parameter (which is here a stochastic process) determining the degree (and level) of this tradeoff. The decision interval is again $[0, t_f]$, and $\beta \geq 0$ is another discount factor.

We will now study the two noncooperative equilibrium solution concepts associated with the game, namely the *Nash* and the *Stackelberg* equilibria (cf. Başar and Olsder, 1982). Recall that a policy pair (γ^N, δ^N) is in Nash equilibrium if

$$\begin{aligned} J^A(\gamma^N,\delta^N) &\leq J^A(\gamma^N,\delta), \qquad \forall \delta \in \Delta \\ J^P(\gamma^N,\delta^N) &\geq J^P(\gamma,\delta^N), \qquad \forall \gamma \in \Gamma. \end{aligned} \tag{7}$$

On the other hand, (γ^s, δ^s) are in Stackelberg equilibrium, with P as leader, if

$$J^P(\gamma^s,\delta^s) \geq J^P(\gamma, T(\gamma)), \qquad \forall \gamma \in \Gamma \tag{8a}$$

where T is defined by the minimization problem (assuming existence of a unique solution – – which turns out to be the case here)

$$J^A(\gamma, T(\gamma)) \leq J^A(\gamma, \delta), \qquad \forall \delta \in \Gamma \qquad \forall \gamma \in \Gamma \tag{8b}$$

and

$$\delta^s = T(\gamma^s). \tag{8c}$$

Note that the (stochastic) information available to the players is asymmetric, making both the Nash and the Stackelberg games nonstandard stochastic differential game problems.

3 The Stackelberg Solution

Toward obtaining the Stackelberg solution, with P as the leader, we first note that for every fixed $\gamma \in \Gamma$, and with the mapping $T : \Gamma \to \Delta$ chosen as

$$T(\gamma)_t(m_0^t) = E[\gamma_t(x_0^t; m_0^t)|m_0^t] =: \hat{m}_t^A \tag{9a}$$

we have

$$\begin{aligned} J^A(\gamma, \delta) &= E\left\{\int_0^{t_f}(m_t^P - \hat{m}_t^A + \hat{m}_t^A - m_t^A)^2 e^{-\tilde{\beta}t}dt\right\} \\ &\equiv E\left\{\int_0^{t_f}(e_t)^2 e^{-\tilde{\beta}t}dt\right\} + E\left\{\int_0^{t}(\hat{m}_t^A - m_t^A)^2 e^{-\tilde{\beta}t}dt\right\} \end{aligned} \tag{9b}$$

where

$$e_t := m_t^P - \hat{m}_t^A \tag{9c}$$

and the cross term has been cancelled because e_t is orthogonal to any random variable measurable with respect to the sigma field generated by $\{m_s\}_{0 \leq s < t}$. Now, since the first term is independent of $\{m_t^A\}$, it follows that the unique response of A to the announced policy γ by P is determined by the mapping T.

Hence, to complete the solution, we have to maximize $J^P(\gamma, T(\gamma))$, over $\gamma \in \Gamma$, where

$$
\begin{aligned}
J^P(\gamma, T(\gamma)) &= E\left\{\int_0^{t_f}[x_t(m_t^P - E[m_t^P|m_0^t]) - \tfrac{1}{2}(m_t^P)^2]e^{-\beta t}dt\right\} \\
&\equiv E\left\{\int_0^{t_f}[(x_t - E[x_t|m_0^t])m_t^P - \tfrac{1}{2}(m_t^P)^2]e^{-\beta t}dt\right\}
\end{aligned}
\tag{10}
$$

where in going from the first to the second line we have used a standard property of conditional expectation. In view of Başar (1988), let us first restrict γ, to the structural form[1]

$$\gamma_t(x_0^t, m_0^t) = M(t)\,[x_t - E[x_t|m_s, s < t]] \tag{11}$$

where $M(\cdot)$ is some (Borel) function defined on $[t_0, t_f]$. Then, the conditional mean

$$\hat{x}_t := E[x_t|m_s, s < t]$$

is generated by the Kalman Filter given below, for each fixed, say piecewise continuous, $M(\cdot)$:

$$d\hat{x}_t = -(1-\rho)\hat{x}_t dt + A(1-\rho)dt + K dm_t; \hat{x}_0 = \bar{x}_0 \tag{12a}$$

$$K(t) = M(t)\Sigma(t)/\sigma_w^2 \tag{12b}$$

$$\dot{\Sigma}(t) = -2(1-\rho)\Sigma(t) + \sigma_v^2 - \frac{M(t)^2\Sigma(t)^2}{\sigma_w^2}; \quad \Sigma(0) = \sigma_0^2, \tag{12c}$$

under which the maximization problem faced by P reduces to the (deterministic) optimal control problem

$$\max_{\{M(t)\}_{t\geq 0}} \int_0^{t_f} e^{-\beta t}[M(t) - \tfrac{1}{2}M(t)^2]\Sigma(t)dt \tag{13a}$$

subject to

$$\dot{\Sigma} = -2(1-\rho)\Sigma + \sigma_v^2 - M^2\Sigma^2/\sigma_w^2; \quad \Sigma(0) = \sigma_0^2. \tag{13b}$$

Note that, viewed as an optimal control problem, the state equation here is the error variance of the Kalman filter, which is affected by the control variable $M(\cdot)$; furthermore, the objective function to be maximized is not jointly quadratic in M and Σ.

[1] It is shown in the Appendix that this structure brings in no loss of generality in the general class of affine policies.

We use the *maximum principle* to solve this problem. Defining the Hamiltonian:

$$H(M,\Sigma;\lambda) := e^{-\beta t}[M - \tfrac{1}{2}M^2]\Sigma + \lambda\left[-2(1-\rho)\Sigma + \sigma_v^2 - \tfrac{M^2\Sigma^2}{\sigma_w^2}\right] \tag{14a}$$

where $\lambda(t)$, $t \geq 0$, is the co-state variable, the maximizing M should solve

$$\max_M H(M,\Sigma;\lambda) \tag{14b}$$

with the differential equation for the co-state variable being

$$\left.\begin{aligned} &\dot{\lambda} = -H_\Sigma = e^{-\beta t}[\tfrac{1}{2}M^2 - M] + 2(1-\rho)\lambda + 2M^2\Sigma\lambda/\sigma_w^2 \\ &\lambda(t_f) = 0. \end{aligned}\right\} \tag{15}$$

A sufficient condition for M to maximize H is

$$H_M = e^{-\beta t}(1-M)\Sigma - 2M\lambda\Sigma^2/\sigma_w^2 = 0 \tag{16a}$$

$$H_{MM} = -e^{-\beta t}\Sigma - 2\lambda\Sigma^2/\sigma_w^2 < 0 \tag{16b}$$

where (since $\Sigma(t) > 0$) the negativity condition is satisfied if $\lambda \geq 0$, which is indeed the case in a neighborhood of the terminal time t_f. Then, solving for M from the condition $H_M = 0$, we obtain

$$\begin{aligned} M(t) &= e^{-\beta t}\Sigma(t)/[e^{-\beta t}\Sigma(t) + 2\lambda(t)\Sigma(t)^2/\sigma_w^2] \\ &\equiv 1/(1 + 2\lambda(t)\Sigma(t)e^{-\beta t}/\sigma_w^2) \end{aligned} \tag{17}$$

which shows that the gain $M(\cdot)$ satisfies the bounds $0 < M(t) \leq 1$.

Let $\Sigma' := \Sigma/\sigma_w^2$, $\rho' := 1-\rho$, $r := \sigma_v^2/\sigma_w^2$, $\lambda' := \lambda e^{\beta t}$. Then, the set of equations to be solved for the optimal solution can be written in more compact form as the following two-point boundary value problem:

$$\left.\begin{aligned} &M(t) = 1/(1 + 2\lambda'(t)\Sigma'(t)) \\ &\dot{\lambda}' = -\tfrac{1}{2}M^2 + (2\rho' + \beta)\lambda'; \quad \lambda'(t_f) = 0 \\ &\dot{\Sigma}' = -2\rho'\Sigma' + r - M^2\Sigma'^2; \quad \Sigma'(0) = \sigma_0^2/\sigma_w^2 \\ &\lambda'(t)\Sigma'(t) > -\tfrac{1}{2}. \end{aligned}\right\} \tag{18}$$

At steady state (i.e., as $t_f \to \infty$) these differential equations reduce to the algebraic equations

$$\left.\begin{array}{l} \bar{M} = 1/(1+2\bar{\lambda}\bar{\Sigma}) \\ 2\bar{\lambda}(2\rho' + \beta)(1+2\bar{\lambda}\bar{\Sigma})^2 = 1 \\ 2\rho'\bar{\Sigma} + 2(2\rho'+\beta)\bar{\lambda}\bar{\Sigma}^2 = r \\ \bar{\lambda}\bar{\Sigma} > -\frac{1}{2}. \end{array}\right\} \quad (19)$$

which can only be solved numerically. Here,

$$\bar{M} = \lim_{t_f\to\infty} M(t;t_f); \quad \bar{\Sigma} = \lim_{t_f\to\infty} \Sigma'(t;t_f); \quad \bar{\lambda} = \lim_{t_f\to\infty} \lambda'(t;t_f).$$

Hence, the steady-state Stackelberg policy of P is

$$\left.\begin{array}{rcl} m_t^P & = & \gamma_t(x_0^t, m_0^t) = \bar{M}[x_t - \hat{x}_t] \\ d\hat{x}_t & = & -\rho'\hat{x}_t dt + A\rho' + \bar{M}\bar{\Sigma} dm_t; \quad \hat{x}_0 = \bar{x}_0 \end{array}\right\} \quad (20)$$

which indicates that the inflationary bias ($E[m_t^P]$) is zero. We now summarize these results in the following theorem:

Theorem 1:

(i) Let there exist a solution to the two-point boundary value problem (18). Then, the stochastic monetary policy and inflation model of Section 2, viewed as a stochastic differential game, admits a Stackelberg solution given by

$$\begin{array}{rcl} \gamma_t^s(x_t, m_0^t) & = & M(t)[x_t - \hat{x}_t] \\ \delta_t^s(m_0^t) & = & E[\gamma_t^s(x_t, m_0^t) \mid m_0^t] = 0, \end{array}$$

where $\hat{x}_t$ is defined by (12).

(ii) If, furthermore, the set of the algebraic equations (19) admits a valid solution, then (20) constitutes a steady-state Stackelberg policy for the policy maker. □

$\rho'\backslash r$	0.5	1	2
0.1	1.14 0.45	1.97 0.39	3.39 0.33
0.2	0.74 0.59	1.31 0.52	2.28 0.45
0.5	0.39 0.79	0.72 0.72	1.29 0.64

Table 1: Estimation error variance ($\bar{\Sigma}$) and the gain in monetary policy ($\bar{M}$) in the Stackelberg game. (In each entry, $\bar{\Sigma}$ is the first figure, and $\bar{M}$ is the second.)

We now provide some numerical values for the solution of (19) (i.e., for $\bar{\Sigma}$ and $\bar{M}$), for the case $\beta = 0$ (no discount factor) and for different values of r and ρ'. These are given in the table below. In each case the solution has been found to be *unique*.

We conclude this section by noting the following consistent pattern through all ranges of the parameters ρ' and r:

$$\begin{array}{lll} \bar{M} & \downarrow r & \uparrow \rho' \\ \bar{\Sigma} & \uparrow r & \downarrow \rho' \\ \bar{M}^2\bar{\Sigma} & \uparrow r & \uparrow \rho'. \end{array}$$

In words, the monetary gain decreases with increasing level of uncertainty in the preference parameter (through driving process variance σ_v^2) and with decreasing measurement uncertainty (σ_w^2) and with decreasing correlation (ρ') between the future and past values of the preference parameter. On the other hand, the estimation error variance ($\bar{\Sigma}$) shows exactly the opposite behavior. The product, $\bar{M}^2\bar{\Sigma}$, represents the variance of the monetary policy around the zero inflationary bias, and it increases with increasing σ_v^2 and ρ', but decreases with increasing σ_w^2.

4 The Nash Solution

For the Nash game, let us first take P's policy to be in the structural form

$$\gamma_t^P(x_t) = M(t)x_t + k(t), \qquad t \geq 0, \tag{21}$$

where $M(\cdot)$ and $k(\cdot)$ are piecewise continuous functions, under which the δ that minimizes J^A is

$$\hat{x}_t := E[x_t | m_0^t], \tag{22a}$$

given by the solution of the stochastic differential equation

$$d\hat{x}_t = -\rho'\hat{x}_t dt + A\rho' dt + K[dm_t - M\hat{x}_t dt - k dt]; \quad \hat{x}_0 = \bar{x}_0. \tag{22b}$$

Here K is the Kalman gain defined earlier (by (12b)), depending explicitly on M.

We now fix the structure of δ, as given above by (22), and consider the maximization of $J^P(\gamma, \delta)$ over $\gamma \in \Gamma$. This is a stochastic control problem with objective function

$$J = E\left\{\int_0^{t_f} e^{-\beta t}[x_t(m_t^P - \xi_t) - \tfrac{1}{2}(m_t^P)^2]dt\right\} \tag{23}$$

and state equation

$$d\xi_t = -(\rho' + KM)\xi_t dt + (A\rho' - Kk)dt + Km_t^P dt + K dw_t; \quad \xi_0 = \bar{x}_0 \tag{24}$$

where m_t^P is the control variable, which is allowed to depend on x_t, as well as ξ_0^t. Note that here K and M are taken as given functions, and their choice is not considered to be part of the optimization problem. We further note that even though x_t is given as the solution of a (linear) stochastic differential equation, its specific form is irrelevant here since the dynamics are not driven by control. This makes the above problem a rather standard LQG control problem, which admits a certainty equivalent optimal controller. Using a dynamic programming approach, we first write down the associated Hamilton-Jacobi-Bellman equation:

$$\begin{aligned} -V_t(t,\xi) &= V_\xi(t,\xi)[-\rho'\xi + A\rho' - KM\xi - Kk] + \tfrac{1}{2}[V_\xi(t,\xi)]^2 K\sigma_w^2 + x_t\xi e^{-\beta t} \\ &+ \tfrac{1}{2}[x_t - Ke^{\beta t}V_\xi(t,\xi)]^2 e^{-\beta t} - x_t[x_t - Ke^{\beta t}V_\xi(t,\xi)]e^{-\beta t} \\ &+ KV_\xi(t,\xi)[x_t - Ke^{\beta t}V_\xi(t,\xi)]; \quad V(t_f,\xi) = 0 \end{aligned}$$

This PDE admits a unique solution, linear in ξ_t:

$$V(t,\xi_t) = q(t)\xi_t + r(t), \qquad t \geq 0 \tag{25a}$$

where q satisfies the x_t-driven linear differential equation

$$\dot{q} = (\rho' + KM)q + x_t e^{-\beta t}; \quad q(t_f) = 0, \tag{25b}$$

and r satisfies another differential equation which we do not give here since its specific form is irrelevant to the optimal solution. The maximizing m_t^P is unique, and is given by

$$m_t^P = x_t - Ke^{\beta t}V_\xi(t,\xi_t) \equiv x_t - Ke^{\beta t}q(t). \tag{26}$$

At this point, it is convenient to introduce

$$q(t) = \alpha(t)x_t + p(t) \tag{27a}$$

where α and p are independent of x, and are the unique solutions of the linear equations

$$\begin{aligned} \dot{\alpha} &= (2\rho' + KM)\alpha + e^{-\beta t}; \quad \alpha(t_f) = 0 \\ \dot{p} &= (\rho' + KM)p + \alpha A\rho'; \quad p(t_f) = 0. \end{aligned}$$

Further letting

$$\alpha' := \alpha e^{\beta t}; \quad p' := pe^{\beta t}, \tag{27b}$$

we arrive at

$$\dot{\alpha}' = (2\rho' + KM + \beta)\alpha' + 1, \quad \alpha'(t_f) = 0 \tag{28a}$$

$$\dot{p}' = (\rho' + KM + \beta)p + \alpha' A \rho'; \quad p'(t_f) = 0. \tag{28b}$$

in terms of which the unique solution to the stochastic control problem (23)–(24) is expressed as

$$m_t^P = [1 - \alpha'(t)K(t)]x_t - K(t)p'(t). \tag{29}$$

Now the important observation is that this is in the same form as the policy we started with in this section, implying that we have *structural consistency.* To complete the derivation of the Nash solution, we also have to require consistency in the coefficient values, which dictates the relationships

$$M(t) = 1 - \alpha'(t)K(t); \quad k(t) = -K(t)p'(t) \tag{30a}$$

where

$$K(t) = \Sigma'(t)M(t) \tag{30b}$$

$$\dot{\Sigma}' = -2\rho'\Sigma' + r - M^2\Sigma'^2; \quad \Sigma'(0) = \sigma_0^2/\sigma_w^2. \tag{30c}$$

Since M here will in general be different from the one obtained in the Stackelberg game, the error variance, Σ, and the Kalman gain, K, will also be generally different. What is also different in this case, from the Stackelberg game, is the presence of *nonzero* inflationary bias:

$$E[m_t^P] = M(t)E[x_t] + k(t) \neq 0. \tag{31}$$

Again we have a two-point boundary value problem (involving α' and Σ'), which is further coupled by the consistency relationship

$$M(t) = 1 - \alpha'(t)\Sigma'(t)M(t)$$

$$\Leftrightarrow \tag{32}$$

$$M(t) = 1/[1 + \alpha'(t)\Sigma'(t)]$$

which replaces a similar relationship [(17)] that existed in the case of the Stackelberg solution (compare α' with $2\lambda'$).

At steady state (i.e., as $t_f \to \infty$), the limiting solutions of these equations, denoted by "bar"s and without "prime"s, satisfy

$$\bar{M} = 1/[1 + \bar{\alpha}\bar{\Sigma}] \tag{33a}$$

$$(2\rho' + \bar{\Sigma}\bar{M}^2 + \beta)\bar{\alpha} + 1 = 0 \tag{33b}$$

$$-2\rho'\bar{\Sigma} + r - \bar{\Sigma}^2\bar{M}^2 = 0 \tag{33c}$$

$$-(\rho' + \bar{\Sigma}\bar{M}^2 + \beta)\bar{k} + \bar{\alpha}A\rho'\bar{\Sigma}\bar{M} = 0. \tag{33d}$$

In terms of these limiting solutions, the steady state Nash policy[2] of P is

$$m_t^P = \gamma(x_t) = \bar{M}x_t + \bar{k} \tag{34}$$

and the one for decision maker A is

$$\delta_t = \hat{x}_t \tag{35a}$$

$$d\hat{x}_t = -\rho'\hat{x}_t dt + A\rho' dt + \bar{\Sigma}\bar{M}[dm_t - \bar{M}\hat{x}_t dt - \bar{k}dt]; \quad \hat{x}_0 = \bar{x}_0. \tag{35b}$$

The steady state inflationary bias is given by

$$E[m_t^P] = \bar{M}A + \bar{k},$$

which is generally nonzero, as mentioned earlier. We now summarize these results in the following theorem.

[2]Using a discretization technique and a recursive derivation as in Başar and Salmon (1989), it can actually be shown that this Nash solution is unique.

Theorem 2:

(i) Let there exist a solution to the two-point boundary value problem (28), (30). Then, the stochastic monetary policy and inflation model of Section 2, viewed as a differential game, admits a Nash solution given by

$$\begin{aligned} \gamma_t^N(x_t) &= [1 - \alpha'(t)K(t)]x_t - K(t)p'(t) \\ \delta_t^N(m_0^t) &= \hat{x}_t \end{aligned}$$

where $\hat{x}_t$ is generated by (22b).

(ii) If, furthermore, the algebraic equations (33a)–(33d) admit a solution set, then (35) constitutes a steady-state Nash equilibrium for the differential game. □

The counterpart of Table 1 can be generated in this case, leading (for $\beta = 0$) to Table 2 given below, where again all entries have been found to be unique. A comparison with Table 1 immediately reveals the important feature: $\bar{\Sigma}^{\text{NASH}} < \bar{\Sigma}^{\text{STACK}}$, $\bar{M}^{\text{NASH}} > \bar{M}^{\text{STACK}}$, which appears to be a consistent pattern.

$\rho' \backslash r$	0.5	1	2
0.1	0.377 1.769	0.518 1.825	0.731 1.872
0.2	0.371 1.597	0.526 1.689	0.739 1.765
0.5	0.373 1.295	0.500 1.414	0.735 1.529

Table 2: Estimation error variance ($\bar{\Sigma}$) and the gain in monetary policy ($\bar{M}$) in the Nash game. (In each entry, $\bar{\Sigma}$ is the first figure, and $\bar{M}$ is the second.)

5 Appendix

Let the policy of P be any affine function of the current value of x_t and the current and past values of m_s; then, γ_t can be written as

$$\gamma_t(x_t, m_0^t) = M(t)[x_t - E[x_t \mid m_s, s \le t]] + \ell(m_s, s \le t) \qquad (*)$$

where ℓ is a general affine function of m_0^t. Let the first term above be denoted by $\tilde{u}_t$ and the second term by $\tilde{\ell}_t$, both of which are random variables (for each t). Furthermore $\tilde{u}_t$ and $\tilde{\ell}_t$ are uncorrelated, since the estimation error

$$e_t := x_t - E[x_t \mid m_s, s \le t]$$

is orthogonal to any affine function of m_0^t.

Now, picking up from (6); using (9a), we obtain, for a general affine γ,

$$\begin{aligned} J^P(\gamma, T(\gamma)) &= E\left\{\int_0^{t_f}[x_t(\tilde{u}_t - E[\tilde{u}_t \mid m_0^t]) - \tfrac{1}{2}\tilde{u}_t^2 - \tfrac{1}{2}\tilde{\ell}_t^2]e^{-\beta t}dt\right\} \\ &\le E\left\{\int_0^{t_f}[(x_t - E[x_t \mid m_0^t])\tilde{u}_t - \tfrac{1}{2}\tilde{u}_t^2]e^{-\beta t}dt\right\} \end{aligned}$$

where the inequality follows since $\tilde{\ell}_t^2 \ge 0$ a.s. This last expression is the same as (10), with m_t^P replaced by (11), since the σ-field generated by $(*)$ is invariant of ℓ.

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